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Nonlinear analysis and complementarity theory

George Isac

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Abstract We present in this paper a short survey of some recent interactions between Nonlinear Analysis and Nonlinear Complementarity. Considering the new relations between Nonlinear Analysis and Complementarity Theory, put in evidence in this paper, we define several open research subjects profitable to both domains.

Keywords Nonlinear analysis · Nonlinear complementarity problems

1 Introduction

The main goal of this paper is to present some new interactions between Nonlinear Analysis and Nonlinear Complementarity. The new interactions will be used to define some open subjects in the sense to stimulate new developments in Nonlinear Analysis and in Complementarity Theory. The concept of complementarity is synonymous with the concept of equilibrium, not only in the physical sense, but also int the economical sense [13,15,23]. Essentially, the Complementarity Theory is represented by a wide class of mathematical models, and it unifies many problems in field such as *mathematical programming, game theory, economics, the study of equilibrium of traffic flows, mechanics, elasticity, the theory of fluid flow through a semi-permeable membrane, maximizing oil production, the study of contact with friction*, and recently in *robotics* [11,13,15,18,23].

The first complementarity problem was defined about 41 years ago. Now, the number of papers published on this subject is much more than 1000. It is well known that Complementarity Theory has two parts: the Linear Complementarity Theory and the Nonlinear Complementarity Theory. This paper is devoted to Nonlinear Complementarity Theory and its relation with Nonlinear Analysis. Certainly, new developments on this subject will be welcomed for both domains.

G. Isac (✉)
Department of Mathematics, Royal Military College of Canada, Kingston, ON, Canada K7K 7B4
e-mail: isac-g@rmc.ca

2 Preliminaries

We denote by $\mathbb{R}$ the real field , by $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ the n-dimensional Euclidean space and by $(H, \langle\cdot,\cdot\rangle)$ an arbitrary Hilbert space. We recall that $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$. As it is well known, $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ is an n-dimensional Hilbert space. We denote also a closed convex cone in a Hilbert space H by $\mathbb{K}$ if it is a closed set that satisfies the following conditions:

(1) $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$; and
(2) $\lambda\mathbb{K} \subseteq \mathbb{K}$, for any $\lambda \in \mathbb{R}_+$.

If in addition $\mathbb{K}$ is such that $\mathbb{K} \bigcap (-\mathbb{K}) = \{0\}$, we say that $\mathbb{K}$ is a *pointed* convex cone. If $\mathbb{K}$ is given, its dual is, by definition, $\mathbb{K}^* = \{y \in H | \langle y, x\rangle \geq 0, \text{ for all } x \in \mathbb{K}\}$. If $\mathbb{K}$ is a closed cone in H, we denote by $\mathrm{P}_{\mathbb{K}}$ the metric projection onto $\mathbb{K}$ (which is well defined), that is, for any $x \in H$, $\mathrm{P}_{\mathbb{K}}(x)$ is the unique element in $\mathbb{K}$ such that

$$\|x - \mathrm{P}_{\mathbb{K}}\| \leq \|x - y\|, \text{ for all } y \in \mathbb{K}.$$

We suppose known the properties of $\mathrm{P}_{\mathbb{K}}$ given, for example, in [15] and [11].

Given a closed pointed convex cone $\mathbb{K} \subset H$ and two mappings $f, g : H \to H$, the Nonlinear Complementarity Problem defined by f and $\mathbb{K}$ is:

$$NCP(f, \mathbb{K}) : \begin{cases} \text{find } x_* \in \mathbb{K} \text{ such that} \\ f(x_*) \in \mathbb{K}^* \text{ and } \langle x_*, f(x_*)\rangle = 0, \end{cases}$$

and the Nonlinear Implicit Complementarity Problem defined by f, g and $\mathbb{K}$ is:

$$NICP(f, g, \mathbb{K}) : \begin{cases} \text{find } x_* \in H \text{ such that} \\ g(x_*) \in \mathbb{K}, f(x_*) \in \mathbb{K}^* \text{ and } \langle g(x_*), f(x_*)\rangle = 0, \end{cases}$$

Finally, we denote by $(E, \|\cdot\|)$ an arbitrary Banach space.

3 Complementarity Theory and its interaction with Nonlinear Analysis

When, in 1990, we decided to write our first book [13] on Nonlinear Complementarity Problems in infinite dimensional Hilbert spaces, our reason was to show this problem to mathematicians working in Applied Mathematics and in Fundamental Mathematics, because Complementarity Theory is interesting and has deep relations with several domains as Linear Algebra, Functional Analysis, Nonlinear Analysis, the theory of Variational Inequalities, Numerical Analysis, Mathematical Modelling, Economics, Optimization and Engineering, amongst others. Until now, I have published on this subject the books [11,13,15,18] and [23].

From the beginning, in the period of 1970–1980, several authors as R.W. Cottle, S. Karamardian, B.C. Eaves, R. Saigal, amongst others, put in evidence natural relations between complementarity problems, variational inequalities and fixed point theory [13,15]. After 1985, in several of our papers we put in evidence the fact that the relation between Nonlinear Complementarity Theory and Fixed Point Theory is in the double sense, i.e. Fixed Point Theory can be used to solve complementarity problems and conversely, Complementarity Theory can be used to find new fixed point theorems. The topological degree, which is the most important mathematical tool used in Nonlinear Analysis, was used in 1972 by

R. Saigal in some of his papers and, after 1985, the topological degree has been used by several authors such as C.D. Ha, M.S. Gowda, R. Sznajder and D. Goeleven among others [15].

In some of our papers published after 1990, we put in evidence the fact that, naturally nonlinear complementarity problems must be studied by the methods developed in Nonlinear Analysis. In this sense, we used the concept of *zero-epi mapping*, a refinement of the concept of topological degree, in the study of nonlinear complementarity problems [14]. P.P. Zabreiko and A. Carbone proved recently that the Skrypnik's topological degree, which is a new development of the classical topological degree for mappings satisfying condition $(S)_+$, has interesting applications to the study of nonlinear complementarity problems, [3,4]. In Nonlinear Analysis it is well known that condition $(S)_+$ introduced by F. Browder [2] is a good substitute to compactness when this is missing. In 1993, in a joint paper with S.M. Gowda [24] we introduced the condition $(S)^1_+$, which is more general then condition $(S)_+$, and we have shown that this condition has interesting applications to Nonlinear Complementarity Theory; other authors considered our condition.

Finally, one of the most important result in Nonlinear Analysis is the *Leray-Schauder Alternative*. This alternative has been used exclusively for solving nonlinear differential or integral equations. Recently, in several of our papers and in our book [16] we proved that the second part of the Leray-Schauder Alternative has interesting applications to Complementarity Theory. This new interaction between the Leray-Schauder Alternative and Complementarity Theory can be a good stimulus for new developments in Complementarity Theory and Nonlinear Analysis. In this paper we will put in evidence new interactions between Nonlinear Complementarity Theory and Nonlinear Analysis.

4 New solvability theorems for nonlinear equations applicable to Complementarity Theory

Let $(E, \|\cdot\|)$ be a Banach space. For any real number $r > 0$ we denote $\overline{B_r} = \{x \in E \,|\, \|x\| \leq r\}$ and $S_r = \{x \in E \,|\, \|x\| = r\}$. We will give a new solvability theorem for a general equation of the form

$$f(x) = 0, \tag{1}$$

where $f : E \to E$ is a completely continuous mapping, i.e., f is continuous and for any bounded set $D \subset E$, $f(D)$ is relatively compact. The starting point of our solvability results is the following classical result.

If $f : [-1, 1] \to \mathbb{R}$ is a continuous mapping and $f(x) \cdot x < 0$ for any $x \in S_1$ then there is an element $x_0 \in \overline{B_1}$ such that $f(x_0) = 0$.

The extension of this result to the n-dimensional Euclidean space was considered in 1940 by C. Miranda who obtained the following result. Consider in $\mathbb{R}^n$ the following sets:

$$J^n = \{(x_1, x_2, \ldots, x_n) \,|\, |x_i| \leq 1, i = 1, 2, \ldots, n\}$$
$$J_i^+ = \{x \in J^n \,|\, x_i = 1\} \text{ (the } n^{th} \text{ face), and}$$
$$J_i^- = \{x \in J^n \,|\, x_i = -1\} \text{ (the opposite face of } J_i^+).$$

Theorem 1 (Miranda) *Let $f = (f_1, f_2, \ldots, f_n) : J^n \to \mathbb{R}^n$ be a continuous function. If for each $i \in \{1, 2, \ldots, n\}$ we have $f_i(x) \geq 0$ for any $x \in J_i^+$ and $f_i(x) \leq 0$ for any $x \in J_i^-$, then there exists $x_* \in J^n$ such that $f(x_*) = 0$.*

Proof The proof of this result appears in [34]. □

We recall that on an arbitrary Banach space $(E, \|\cdot\|)$, at least two interesting semi-inner-products can be defined. A semi-inner-product, as introduced by Lumer [33], is a mapping $[\cdot,\cdot]_\ell : E \times E \to \mathbb{R}$ such that

$$(S_1)\ \ [x+y,z]_\ell = [x,z]_\ell + [y,z]_\ell\,,\ \text{for any } x,y,z \in E,$$
$$(S_2)\ \ [\lambda x,y]_\ell = \lambda\,[x,y]_\ell\,,\ \text{for any } \lambda \in \mathbb{R} \text{ and any } x,y \in E,$$
$$(S_3)\ \ [x,x]_\ell > 0 \text{ for any } x \in E, x \neq 0,$$
$$(S_4)\ \ \left|[x,y]_\ell\right|^2 \leq [x,x]_\ell \cdot [y,y]_\ell\,,\ \text{for any } x,y \in E.$$

Any Banach space can be endowed with a semi-inner-product in Lumer's sense [33]. A semi-inner-product in Deimling's sense is defined by

$$[x,y]_d = \|y\| \lim_{t\to 0_+} \frac{\|y+tx\| - \|y\|}{t}, \text{ for any } x,y \in E.$$

This semi-inner-product is not linear in the first variable, but it is sublinear [21].

Let$(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a mapping.

Definition 1 (*Almost solvability*) We say that the equation $f(x) = 0$ is almost solvable if there exists $r > 0$ such that $0 \in \overline{f(\bar{B}_r)}$.

The almost solvability means that there exists $r > 0$ such that for any $\epsilon > 0$ there exists $x_\epsilon \in \overline{B_r}$ such that $\|f(x_\epsilon)\| < \epsilon$ (i.e. for any $\epsilon > 0$ there exists an ϵ-solution of equation (1). Obviously, if Eq. 1 is solvable, then it is almost solvable but the converse in not generally true.

We will consider a mapping $F : E \times E \to \mathbb{R}$ satisfying the following properties

$$(g_1)\ G(x,x) \geq 0 \text{ for any } x \in S_r \text{ (for a particular } r > 0),$$
$$(g_2)\ G(\lambda x, y) \geq \lambda G(x,y) \text{ for any } \lambda > 0 \text{ and all } x,y \in S_r.$$

Example 1

(1) If $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space, we can take $G(\cdot,\cdot) = \langle\cdot,\cdot\rangle$.
(2) If $(E, \|\cdot\|)$ is a Banach space, then we can take $G(\cdot,\cdot) = [\cdot,\cdot]_\ell$ or $G(\cdot,\cdot) = [\cdot,\cdot]_d$.
(3) Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and $E = \mathcal{C}([0,1], H)$ and $\|x\| = \sup_{t\in[0,1]} \|x(t)\|$ for any $x \in E$. In this case we can take $G(x,y) = \sup_{t\in[0,1]} \langle x(t), y(t)\rangle$ or $G(x,y) = \int_0^1 \langle x(t), y(t)\rangle\, dt$.

□

Theorem 2 (Isac-Avramescu) *Let $(E, \|\cdot\|)$ be a Banach space and $f : E \to E$ a mapping. If the following assumptions are satisfied:*

(1) *f is completely continuous;*
(2) *there exists a mapping $G : E \times E \to \mathbb{R}$ satisfying (g_1) and (g_2) for a particular $r > 0$; and*
(3) *$G(f(x), x) < 0$ for any $x \in S_r$,*

then the equation $f(x) = 0$ is almost solvable in $\overline{B_r}$.

Proof For the proof, the reader is referred to [20,21]. □

Remark 1

(1) Theorem 2 is valid if assumption (3) is replaced with $G(f(x), x) > 0$ for any $x \in S_r$.
(2) If the space is finite dimensional, then the almost solvability implies the solvability.

Theorem 3 (Isac-Avramescu) *Let* $(E, \|\cdot\|)$ *be a Banach space and* $f : E \to E$ *a completely continuous mapping. Suppose that* $G : E \times E \to \mathbb{R}$ *is an inner-product dominated by the norm* $\|\cdot\|$ *of* E *i.e.,* $|G(x, y)| \leq k\|x\|\|y\|$ *for any* $x, y \in E$ *(where* $k > 0$*). Denote by* $M = \sup_{x \in \overline{B_r}} \|f(x)\|$ *for a particular* $r > 0$.

If $\|f(x) - f(y)\| \leq a\|x - y\|$ *for any* $x, y \in \overline{B_r}$*, where* $a > 0$*, and* $G(f(x), x) \leq -c$*, where* $c = rk[ar + m]$*, for any* $x \in S_r$*, then the equation* $f(x) = 0$ *has a solution in* $\overline{B_r}$.

Proof For the proof of this result the reader is referred to [21]. □

Now, we give some applications to nonlinear complementarity problems. First, we consider the n-dimensional Euclidean space $(\mathbb{R}^n, \langle\cdot, \cdot\rangle)$. Let $\mathbb{K} \subset \mathbb{R}^n$ be a closed convex cone and $h : \mathbb{R}^n \to \mathbb{R}^n$ a mapping. We consider the nonlinear complementarity problem $NCP(h, \mathbb{K})$. We have the following result.

Theorem 4 *Lt* $\mathbb{K} \subset \mathbb{R}^n$ *be a closed convex cone and* $h : \mathbb{R}^n \to \mathbb{R}^n$ *a continuous mapping. If there exist two real numbers* $\alpha > 0$ *and* $r > 0$ *such that* $\|x - \alpha h(x)\| < r$ *for any* $x \in \mathbb{R}^n$ *with* $\|x\| = r$*, then the problem* $NCP(h, \mathbb{K})$ *has a solution* x_0 *with* $\|x_0\| \leq r$.

Proof (Sketch) We apply Theorem 2 taking $G(x, y) = \langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$ and $f(x) = \mathrm{P}_{\mathbb{K}}[x - \alpha h(x)] = x$. □

Example 2 If $h(x) = \beta x - g(x)$, where $\beta > 0$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping with the property that there exists $\rho > 0$ such that for any x with $\|x\| > \rho$ we have $\|g(x)\| < r_0$, $r_0 > 0$, then in this case we take $\alpha = \frac{1}{\beta}$ and $r > \max\left\{\frac{1}{\beta}r_0, \rho\right\}$. □

Remark 2 In the case of the linear complementarity problem $LCP(A, b, \mathbb{R}^n_+)$, we can estimate the number r used in Theorem 2 considering the spectrum of the matrix $A_0 = A + A^*$, where A^* is the adjoint matrix of A. In this sense, see the method to estimate the radius of the ball containing all the solutions of the problem $LCP(A, b, \mathbb{R}^n_+)$, developed in [12].

We note the condition $\langle f(x), x\rangle = \langle \mathrm{P}_{\mathbb{K}}[x - \alpha h(x)] - x, x\rangle < 0$ used in Theorem 2 or 4, is equivalent with the following global optimization problem:

$$\begin{cases} \text{find the global max of } \langle \mathrm{P}_{\mathbb{K}}[x - \alpha h(x)] - x, x\rangle \\ \text{when } x \in S_r \end{cases}$$

and if $x_* \in S_r$ is a solution of this program, we must have $\langle \mathrm{P}_{\mathbb{K}}[x_* - \alpha h(x)] - x_*, x_*\rangle < 0$. Now we consider the case of an infinite dimensional Hilbert space. Let $(H, \langle\cdot, \cdot\rangle)$ be an infinite dimensional Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $h : H \to H$ a completely continuous mapping. In this case, the solvability (or the almost solvability) cannot be studied by Theorem 2 or 3 because the mapping $f(x) = \mathrm{P}_{\mathbb{K}}[x - \alpha h(x)] - x$ can not be completely continuous. However, the implicit complementarity problem $INCP(f, g, \mathbb{K})$, where f and g are completely continuous can be studied. In this sense we have the following results □

Theorem 5 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f, g : H \to H$ two completely continuous mappings. Let $G : H \times H \to \mathbb{R}$ be a mapping satisfying conditions (g_1) and (g_2). If there exists $r > 0$ such that $G(\mathrm{P}_{\mathbb{K}}[x - \alpha h(x)] - x, x) < 0$ for all $x \in S_r$, then the problem $INCP(f, g, \mathbb{K})$ is almost solvable in $\overline{B_r}$, that is, for any $\epsilon > 0$ there exists $x_\epsilon \in \overline{B_r}$ satisfying the inequality*

$$\|\mathrm{P}_{\mathbb{K}}[g(x_\epsilon) - f(x_\epsilon)] - g(x_\epsilon)\| < \epsilon.$$

Proof The proof is based on Theorem 2. □

Theorem 6 *If $f, g : H \to H$ are completely continuous mappings and the following assumptions are satisfied:*

(1) *there exists $r > 0$ and $M > 0$ such that $\langle g(x), x\rangle \geq M\|h\|^2$, for any $x \in S_r$,*
(2) *$\|g(x) - f(x)\| < Mr$, for any $x \in S_r$,*

then the problem $INCP(f, g, \mathbb{K})$ is almost solvable in $\overline{B_r}$.

Proof This result is a consequence of Theorem 5. □

Remark 3 If we have to solve the problem $NCP(f, \mathbb{K})$ in an infinite dimensional Hilbert space, we can transform the problem in an implicit complementarity problem of the form $NCP(\mathcal{F}, G, \mathbb{K})$ if f has the form $f(x) - T(x)$, where $T : H \to H$ is a completely continuous mapping, taking $G(x) = \varphi(x)$, and $\mathcal{F}(x) = \varphi(x) - T(\varphi(x))$. The solvability of the problem $INCP(\mathcal{F}, G, \mathbb{K})$ implies the solvability of the problem $NCP(f, \mathbb{K})$. □

If the $INCP(f, g, \mathbb{K})$ is almost solvable in $\overline{B_r}$, then for any $\epsilon > 0$ (eventually very small) there exist $x_\epsilon \in H$ with $\|x_\epsilon\| \leq r$ and u_ϵ with $\|u_\epsilon\| < \epsilon$ such that

$$\begin{cases} g(x_\epsilon) + u_\epsilon \in \mathbb{K} \\ f(x_\epsilon) + u_\epsilon \in \mathbb{K}^* \text{ and} \\ \langle g(x_\epsilon) + u_\epsilon, f(x_\epsilon) + u_\epsilon\rangle = 0. \end{cases} \tag{2}$$

The mappings $f(\cdot) + u_\epsilon$ and $g(\cdot) + u_\epsilon$ may be considered as *small perturbations of f and g*. Considering relation (2) we can say that the almost solvability of the problem $INCP(f, g, \mathbb{K})$ means that for any $\epsilon > 0$ there exist u_ϵ with $\|u_\epsilon\| < \epsilon$ such that the problem $INCP(f(\cdot) + u_\epsilon, g(\cdot) + u_\epsilon, \mathbb{K})$ has a solution x_ϵ with $\|x_\epsilon\| \leq r$.

4.1 Open subjects

Considering the results presented in this section, we define the following open subjects.

(1) New solvability or almost solvability theorems applicable to complementarity problems are necessary.
(2) Given the problem $NCP(f, \mathbb{K})$, where H is an infinite dimensional Hilbert space, $\mathbb{K} \subset H$ is a closed convex cone and $f : H \to H$, we can consider the *normal* operator defined by

$$\mathcal{N}_\alpha(x) = f(\mathrm{P}_{\mathbb{K}}(x)) + \alpha(x - \mathrm{P}_{\mathbb{K}}(x)), \text{ where } \alpha > 0 \text{ and } x \text{ is arbitrary in } H.$$

It is known that $\mathcal{N}_\alpha(x) = 0$ has a solution if and only if the $NCP(f, \mathbb{K})$ has a solution. The operator $\mathcal{N}_\alpha$ was studied for $\alpha = 1$ in [37,38] and [39]. Because the operator $\mathcal{N}_\alpha$ can not be completely continuous in an infinite dimensional Hilbert space, it would be useful to find new solvability theorems for the equation $\mathcal{N}_\alpha(x) = 0$.

(3) Some surjectivity theorems for the operator $\mathcal{N}_\alpha$ can have interesting applications to the study of nonlinear complementarity problems.

5 The notion of Exceptional Family of Elements and a general coercivity condition

In the first period of development of Complementarity Theory, the solvability of nonlinear complementarity problems was studied by the fixed point theory, by some special results of convex analysis and by KKM type theorems. In some papers the *coercivity condition* was also used. After 1992, we considered the idea to find a general coercivity condition applicable to complementarity problems. Finally, we arrived to find the notion of *exceptional family of elements* (EFE) for a continuous function and we used this notion in several papers [16,22,25,29,40]. About the investigation method based on the notion of EFE, the reader is referred to [15,18] and [23]. We now recall some notions, results and the concept of EFE.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $X \subset H$ a non-empty set and $f : H \to H$ a mapping. We denote by $\overline{X}$ the closer of X and by $tialX$ the boundary of X. We say that f is compact on X if $f(X)$ is a relatively compact set. We denote by α the *Kuratowski measure* of non compactness (see the definition in [18]). Let $K \subset H$ be a closed convex cone. We say that f is a *α-condensing mapping* if f is continuous and bounded (i.e. if $B \subset H$ is bounded then $f(B)$ is also bounded), and $\alpha(f(B)) < \alpha(B)$, for all $B \subset H$ with $\alpha(B) > 0$.

We say that f is a *completely continuous field* if $f(x) = x - T(x)$, for any $x \in H$, where $T : H \to H$ is a completely continuous mapping, and we say that f is an *α-condensing field* if T is an α-condensing mapping. We recall the following result.

Theorem 7 (Leray-Schauder Alternative) *Let $D \subset H$ be a convex set, U a subset open in D and such that $0 \in U$. Then each continuous compact mapping $f : \overline{U} \to D$ has at least one of the following properties:*

(1) *f has a fixed point,*
(2) *there is $(x_*, \lambda_*) \in \partial U \times]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Proof A proof is given in [18]. □

Definition 2 We say that a family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$ is and EFE for a continuous mapping $f : H \to H$ with respect to $\mathbb{K}$, if for every real $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the following conditions:

(1) $u_r \in \mathbb{K}^*$,
(2) $\langle u_r, x_r\rangle = 0$
(3) $\|x_r\| \to \infty$ as $r \to \infty$.

This notion is justified by the following result due to G. Isac [16].

Theorem 8 (Alternative) *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f : H \to H$ a mapping. If one of the following conditions is satisfied,*

(1) *f is a completely continuous field,*
(2) *f is an α-condensing field,*

then there exists either a solution to the problem $NCP(f, \mathbb{K})$ or f has an EFE with respect to $\mathbb{K}$.

Corollary 1 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone. If the function $f : H \to H$ is a completely continuous filed, or an α-condensing field without an EFE, with respect to $\mathbb{K}$, then the problem $NCP(f, \mathbb{K})$ has a solution.*

We recall the classical notion of *coercivity* used in the theory of variational inequalities. We say that $f : H \to H$ is *coercive* with respect to $\mathbb{K}$, if there is an element $x_0 \in \mathbb{K}$ such that

$$\lim_{\substack{\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{\langle f(x) - f(x_0), x - x_0\rangle}{\|x - x_0\|} = +\infty$$

We can prove that if f is coercive then it is without an EFE. In [18] are presented several classes of mappings with the property of being without an EFE with respect to convex cones, but these mappings are not coercive. Consequently, we can say that the property of being without EFE is a general coercive condition. A particular case, useful for practical problems, is given by the following condition due to G. Isac.

Definition 3 We say that a mapping $f : H \to H$ satisfies condition (θ) with respect to a convex cone $\mathbb{K} \in H$ if there exists a real number $\rho > 0$ such that for each $x \in \mathbb{K}$ with $\|x\| > \rho$ there exists $y \in \mathbb{K}$ with $\|y\| < \|x\|$ such that $\langle x - y, f(x)\rangle \geq 0$.

Proposition 1 *If $f : H \to H$ satisfies condition (θ) with respect to $\mathbb{K}$, then f is without EFE with respect to $\mathbb{K}$.*

In several of our papers and in our book [18], we proved that several classes of mappings used in Complementarity Theory satisfy condition (θ). Because only coercive mappings satisfy (θ), we can say that condition (θ) is also a kind of coercive condition.

We note that there exists other classes of mappings which are without EFE but which do not satisfy condition (θ). This is the case of mappings satisfying a *feasibility condition*.

We recall that the *strict dual* of $\mathbb{K}$ is

$$\widehat{\mathbb{K}^*} = \left\{y \in H \,\middle|\, \langle y, x\rangle > 0 \text{ for all } x \in \mathbb{K} \setminus \{0\}\right\}$$

For some cones, the strict dual can be empty. Any well-based cone [15] has a non empty strict dual. We say that the problem $NCP(f, \mathbb{K})$ is *strictly feasible* if there exists $x_0 \in \mathbb{K}$ such that $f(x_0) \in \widehat{\mathbb{K}^*}$.

We recall that a mapping $f : H \to H$ is said to be pseudomonotone with respect to $\mathbb{K}$ if for every $x, y \in \mathbb{K}$ we have that $\langle x - y, f(y)\rangle \geq 0$ implies that $\langle x - y, f(x)\rangle \geq 0$. We have the following result.

Theorem 9 (Isac-Kalashnikov) *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed well-based convex cone and $f : H \to H$ a mapping. If the problem $NCP(f, \mathbb{K})$ is strictly feasible then f is without EFE*

Proof A proof of this result in given in [25]. □

Remark 4 There exist other classes of mappings with the property that the feasibility implies the non existence of EFE. In this sense we cite the quasi-P_*-mappings and the $P(\tau, \alpha, \beta)$-mappings defined and studied in [40]. □

We note that the notion of EFE was adapted for complementarity problems defined by set-valued mappings, for implicit complementarity problems and for variational inequalities. The following question naturally arises. *Under what conditions does the solvability of the problem $NCP(f, \mathbb{K})$ imply that f is without EFE with respect to $\mathbb{K}$?* We now cite an interesting results in this sense.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f, g : H \to H$ two mappings. The following definition is due to G. Isac.

Definition 4 We say that f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$ if there exists a real number $\rho < 0$ such that for all $x, y \in \mathbb{K}$ with $\max\{\rho, \|y\|\} < \|x\|$, we have that $\langle x - y, g(y) \rangle \geq 0$ implies $\langle x - y, f(x) \rangle \geq 0$

We have the following result.

Theorem 10 *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f, g : H \to H$ two mappings. If f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$ and the problem $NCP(g, \mathbb{K})$ has a solution, then f is without EFE with respect to $\mathbb{K}$.*

From Theorem 10 we deduce the following two interesting results, applicable to the study of equilibrium of integrated economical systems.

Theorem 11 (Transitivity principle) *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f, g : H \to H$ two mappings. If the following assumptions are satisfied:*

(1) *f is a completely continuous field or an α-condensing field;*
(2) *f is asymptotically g-pseudomonotone with respect to $\mathbb{K}$;*
(3) *the problem $NCP(g, \mathbb{K})$ has a solution,*

then the problem $NCP(f, \mathbb{K})$ has a solution.

Corollary 2 *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a completely continuous field or an α-condensing field, if f is pseudomonotone with respect to $\mathbb{K}$, then the problem $NCP(f, \mathbb{K})$ has a solution if and only if f is without EFE with respect to $\mathbb{K}$.*

Finally, related to the notion of EFE we consider the following problem: *Is it possible to find a necessary and sufficient condition to have that a given function has the property of being without EFE with respect to a given convex cone?* The next results of this section are related to this problem.

For any $r > 0$ we denote $\mathbb{K}_r = \left\{x \in \mathbb{K} \middle| \, \|x\| \leq r\right\}$.

Definition 5 We say that a family of elements $\{x_r\}_{r>0} \subset \mathbb{K}$ is a regular exceptional family of elements (REFE) for f with respect to $\mathbb{K}$, if for every real number $r > 0$ there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the properties:

(1) $u_r \in \mathbb{K}^*$,
(2) $\langle u_r, x_r \rangle = 0$,
(3) $\|x_r\| = r$.

Definition 6 We say that a mapping $f : H \to H$ is REFE-acceptable with respect to $\mathbb{K}$ if either the problem $NCP(f, \mathbb{K})$ has a solution, or the mapping f has a REFE with respect to $\mathbb{K}$.

Remark 5 Obviously, if f is a REFE-acceptable mapping with respect to $\mathbb{K}$ and it is without REFE, then the problem $NCP(f, \mathbb{K})$ has a solution. □

Theorem 12 *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ be a closed convex cone and $f : H \to H$ a mapping such that for any $r > 0$ the mapping $\Psi_r = \mathrm{P}_{\mathbb{K}_r} \circ (I - f)$ has a fixed point, where Ψ_r is considered from $\mathbb{K}_r$ to $\mathbb{K}_r$. Then f is REFE-acceptable with respect to $\mathbb{K}$.*

Proof A proof of this result is given in [28]. □

Example 3 In the n-dimensional Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ any continuous mappings is REFE-acceptable with respect to any closed convex cone. □

Example 4 If $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $\mathbb{K}$ a convex cone with a compact base then any continuous mapping is REFE-acceptable with respect to $\mathbb{K}$. □

Example 5 If $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space and $\mathbb{K} \subset H$ a closed convex cone, then in this case completely continuous fields and α-condensing fields are REFE-acceptable with respect to $\mathbb{K}$. □

The reader can find other examples of REFE-acceptable mappings in the book [18] an in our recent paper [28].

Theorem 13 *Let$(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone and $f : H \to H$ a mapping. A necessary and sufficient condition for the mapping f to have the property of being without REFE with respect to $\mathbb{K}$ s the following: there is a $\rho > 0$ such that for any $x \in \mathbb{K}$ with $\|x\| = \rho$ at least one of the following conditions hold:*

(1) $\langle f(x), x \rangle \geq 0$,
(2) there is a $y \in \mathbb{K}$ *such that* $\rho^2 \langle f(x), y \rangle < \langle x, y \rangle \cdot \langle f(x), x \rangle$.

Proof A proof of this is result is given in [28]. □

5.1 Open subjects

(1) It is interesting and useful to find new classes of mappings having the property of being without EFE or REFE.
(2) It is interesting and useful to find new classes of mappings with the property that feasibility implies the non existence of EFE.
(3) It is interesting to find new existence theorems for the problem $NCP(f, \mathbb{K})$ based on Theorem 13.
(4) It is interesting to find new classes of mappings which are REFE-acceptable with respect to a general closed convex cone in a Hilbert space.
(5) It is interesting to study deeply the asymptotically g-pseudomonotonicity.

6 Asymptotic differentiable fields, scalar differentiability and fixed points theorems on cones applicable to complementarity problems

In this section we present a variant of Krasnoselskii's fixed point theorem on convex cones [31]. Our theorem is based on asymptotic and on scalar differentiability and we apply this theorem to nonlinear complementarity problems. This application can be considered as a stimulus for new developments in the theory of asymptotic scalar differentiability.

Let $(E, \|\cdot\|)$ be a Banach space and let $[\cdot, \cdot]$ be a semi-inner-product in Lumer's sense as defined in Sect. 4 of this paper ($[\cdot, \cdot]_\ell$). If E is a Hilbert space then in this case we have on E only one semi-inner-product which is exactly the inner-product given on E. For some results we need to consider a semi-inner-product $[\cdot, \cdot]$ on e satisfying the supplementary condition

$$(S_5)\ \ [x, \lambda y] = \lambda\, [x, y] \ \text{ for any } x, y \in E \text{ and any } \lambda \in \mathbb{R}.$$

If the semi-inner-product $[\cdot, \cdot]$ is given on E we consider the norm defined by $\|x\|_S = [x, x]^{\frac{1}{2}}$ for any $x \in E$. If for any $x \in E$, $[x, x] = \|x\|^2$, we say that the semi-inner-product is compatible with the norm $\|\cdot\|$ given on E. The operator $i : E \setminus \{0\} \to E \setminus \{0\}$ defined by $i(x) = \frac{x}{[x,x]}$ is called *inversion* (of pole 0) with respect to $[\cdot, \cdot]$. Let $A \subset E$ be a subset such that $0 \in A$ and $A \setminus \{0\}$ is an invariant set with respect to i, i.e. $i(A \setminus \{0\}) = A \setminus \{0\}$. Let $f : A \to E$ be a mapping. The inversion (of pole 0) with respect to $[\cdot, \cdot]$ is

$$\mathcal{I}(f)(x) = \begin{cases} [x, x]\,(f \circ i)(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We note that any closed convex cone $\mathbb{K} \subset E$ is invariant with respect to any semi-inner-product.

Let $\subset E$ be a closed convex cone and $f : E \to E$ a mappings. We say that f is *positive* if $f(\mathbb{K}) \subseteq \mathbb{K}$. We denote by $\mathcal{L}(E, E)$ the set of linear bounded mappings from E to E.

Definition 7 (from Krasnoselskii [31]) We say that a nonlinear mapping $f : E \to E$ is asymptotically linear along $\mathbb{K}$ if there exists $T \in \mathcal{L}(E, E)$ such that

$$\lim_{\substack{\|x\| \to \infty \\ x \in \mathbb{K}}} \frac{\|f(x) - T(x)\|}{\|x\|} = 0$$

If the cone $\mathbb{K}$ is generating, i.e. $E = K - K$, then T satisfying Definition 7 is unique and in this case we say that T is the asymptotic derivative of f along $\mathbb{K}$ and we denote T by $f_{\mathbb{K}}^{\infty}$. When $\mathbb{K} = E$, $f_{\mathbb{K}}^{\infty}$ will be denoted by f_{∞}.

It is known that if f is positive then $f_{\mathbb{K}}^{\infty}$ is positive and if f is completely continuous with respect to $\mathbb{K}$, then so is $f_{\mathbb{K}}^{\infty}$, [1], [31].

Let $G \subseteq E$ be a set which contains at least one non isolated point, $\widetilde{G} \subseteq E$ such that $G \subseteq \widetilde{G}$, $f : \widetilde{G} \to E$ and x_0 a non isolated point of G. The following definition is due to S.Z. Nemèth [26,27,29,35,36].

Definition 8 The limit

$$\underline{f}^{\#,G}(x_0) = \liminf_{\substack{x \to x_0 \\ x \in G}} \frac{[f(x) - f(x_0), x - x_0]}{\|x - x_0\|_S^2}$$

is called the *lower scalar derivative* of f at x_0 along G, with respect to $[\cdot,\cdot]$. Replacing lim inf by lim sup in the above, we obtain the *upper scalar derivative* denoted by $\overline{f}^{\#,G}(x_0)$.

Definition 9 We say that $T \in \mathcal{L}(E, E)$ is a scalar asymptotic derivative of f along the cone $\mathbb{K} \subset E$, with respect to $[\cdot,\cdot]$ if

$$\liminf_{\substack{\|x\|\to\infty \\ x\in\mathbb{K}}} \frac{[f(x) - T(x), x]}{\|x\|_S^2} \le 0$$

We denote T by $f'_{S,\mathbb{K}}(\infty)$.

We cite the following results proved in [29].

Proposition 2 *If T is a scalar asymptotic derivative of f with respect to $[\cdot,\cdot]$ (along $\mathbb{K}$), then for any $c > 0$ the mapping $T + cI$ i also a scalar asymptotic derivative of f with respect to $[\cdot,\cdot]$.*

Proposition 3 *If the semi-inner-product $[\cdot,\cdot]$ is compatible with the norm $\|\cdot\|$ of E and if $T \in \mathcal{L}(E, E)$ is the asymptotic derivative of f with respect to $\mathbb{K}$, then T is a scalar asymptotic derivative of f.*

Now, we give some ideas about the computation of asymptotic and of scalar derivatives.

(1) Generally, we can compute the asymptotic derivative using the definition and the particularity of the operator. This is the case of integral operators (Hammerstein or Urysohn integral operators). For this case, the reader is referred to [30] and [31].
(2) Some results obtained in the theory of Hyers-Ulam stability of mappings can be used do compute the asymptotic derivative of a mapping. We cite only a recent result in this sense. Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying the properties
    (a) $\lim_{t\to\infty} \frac{\Psi(t)}{t} = 0$;
    (b) $\Psi(st) \le \Psi(s)\Psi(t)$, for any $s, t \in \mathbb{R}_+$;
    (c) $\Psi(t) < t$, for all $t > 1$.
    And let $\mathcal{F}(\Psi)$ be the family of all such functions Ψ, and let $\mathcal{P}(\Psi)$ be the convex cone generated by $\mathcal{F}(\Psi)$.

    Theorem 14 (Isac) *Let $f : E \to E$ be a continuous mapping and $\Psi \in \mathcal{P}(\Psi)$ such that $\Psi = \sum_{i=1}^m a_i\Psi_i$, $a_i > 0$ and $\Psi_i \in \mathcal{F}(\Psi)$ for any i. If f is Ψ-additive, i.e. there exists $\theta > 0$ such that $\|f(x + y) - f(x) - f(y)\| \le \theta\,[\Psi(\|x\|) + \Psi(\|y\|)]$ for any $x, y \in E$, and if $f(S)$ is bounded, where $S = \left\{x \in E \,\middle|\, \|x\| = 1\right\}$, then f has an asymptotic derivative f_∞ (along E) and $f_\infty(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, for any $x \in E$.*

    Proof A proof of this result is given in [17]. □

    The scalar derivative offers us also a possibility to decide if a nonlinear mapping is scalarly asymptotic differentiable. In this sense we cite only the following results proved in [29].

    Theorem 15 (Isac-Németh) *If the semi-inner product $[\cdot,\cdot]$ is compatible with the norm $\|\cdot\|$ of E, then $T \in Ł(E, E)$ is a scalar asymptotic derivative of f with respect to $[\cdot,\cdot]$ along a closed convex cone $\mathbb{K} \in E$, if and only if the upper scalar derivative of h in 0 is non-positive, i.e., $\overline{h}^{\#}(0) \le 0$, where $h = \mathcal{I}(f - T \circ j)$ and $j : \mathbb{K} \to E$ is the embedding of $\mathbb{K}$ in E.*

Theorem 16 (Isac-Németh) *If the semi-inner product* $[\cdot,\cdot]$ *is compatible with the norm* $\|\cdot\|$ *and* $\overline{\mathcal{I}(f)^{\#}(0)} < +\infty$, *then* f *is scalarly asymptotic differentiable with respect to* $[\cdot,\cdot]$ *and* $T = \overline{\mathcal{I}(f)^{\#}(0)} \cdot I$, *is a scalar asymptotic derivative of* f *with respect to* $[\cdot,\cdot]$, *where* $I : E \to E$ *is the identity operator.*

Now, we give some fixed point theorems with respect to a convex cone, applicable to complementarity problems. First, we recall the Krasnoselskii fixed point theorem, with respect to a convex cone.

Theorem 17 (Krasnoselskii) *Let* $(E, \|\cdot\|)$ *be a Banach space,* $\mathbb{K} \subset E$ *a generating closed pointed convex cone, and* $f : \mathbb{K} \to \mathbb{K}$ *a completely continuous mapping. If* f *is asymptotically differentiable and* $f^{\infty}_{\mathbb{K}}$ *is its asymptotic derivative with* $r(f^{\infty}_{\mathbb{K}}) < 1$ *(where* $r(\cdot)$ *is the spectral radius), then* f *has a fixed point.*

Proof A proof of this result is given in [1] and [31]. □

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a closed convex cone.

Definition 10 We say that $f : \mathbb{K} \to \mathbb{K}$ is scalarly compact if for any sequence $\{x\}_{n\in\mathbb{N}} \subset \mathbb{K}$, weakly convergent to an element $x_* \in \mathbb{K}$, there exists a subsequence $\left\{x_{n_k}\right\}_{k\in\mathbb{N}}$ such that

$$\limsup_{\mathbb{K}\to\infty} \left\langle x_{n_k} - x_*, f(x_{n_k}\right\rangle \leq 0.$$

Remark 6 Any completely continuous mapping is scalarly compact. □

Our fixed point theorem with respect to a convex cone is the following. We denote by $\underset{(\mathbb{K}^*)}{\leq}$ the ordering defined by $\mathbb{K}^*$, i.e. $h_1, h_2 : \mathbb{K} \to \mathbb{K}$, $h_1 \underset{(\mathbb{K}^*)}{\leq} h_2$ means $h_i(x) \leq h_2(x)$ for any $x \in \mathbb{K}$.

Theorem 18 (Isac) *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a pointed closed convex cone and* $f : \mathbb{K} \to \mathbb{K}$ *a mapping. If the following assumptions are satisfied:*

(1) f *is demi-continuous,*
(2) f *is scalarly compact,*
(3) *there exists a scalar asymptotic derivable mapping* $f_0 : \mathbb{K} \to E$ *such that* $f \underset{(\mathbb{K}^*)}{\leq} f_0$ *and* $\left\| f_{0S,\mathbb{K}}(\infty)\right\| < 1$,

then f *has a fixed point in* $\mathbb{K}$.

Proof For a proof of this result the reader is referred to [17]. □

Remark 7 We note that Theorem 18 is more flexible for applications then Theorem 17. □

Corollary 3 *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $\mathbb{K} \subset H$ *a pointed closed convex cone and* $f : \mathbb{K} \to \mathbb{K}$ *a mapping. If the following assumptions are satisfied*

(1) *f is demi-continuous,*
(2) *f is scalarly compact,*
(3) *f has a scalar asymptotic derivative $f_{S,K}(\infty)$ and $\left\| f_{S,K}(\infty) \right\| < 1$,*

then f has a fixed point in $\mathbb{K}$.

Theorem 19 (Isac-Németh) *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a generating closed pointed convex cone and $f : \mathbb{K} \to \mathbb{K}$ a mapping. If the following assumptions are satisfied*

(1) *f is demi-continuous,*
(2) *f is scalarly compact,*
(3) *there exists a mapping $f_0 : \mathbb{K} \to H$, such that $f \underset{(\mathbb{K}^*)}{\leq} f_0$ and $\mathcal{I}(f_0)(0) < 1$,*

then f has a fixed point in $\mathbb{K}$.

Proof We can show (see [29]) that f_0 has a scalar asymptotic derivative $f_{0S,\mathbb{K}}(\infty)$ with $\left\| f_{0S,\mathbb{K}}(\infty) \right\| < 1$ □

We now give an application of Theorem 18 to complementarity problems.

Theorem 20 *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\mathbb{K} \subset H$ a pointed closed convex cone and $f : H \to h$ a mapping such that there exist two continuous mappings $A, T : H \to H$ such that $f(x) = x - A(x) - T(x)$, for any $x \in H$. If the following assumptions are satisfied:*

(1) $\lim\limits_{\substack{\|x\| \to \infty \\ x \in \mathbb{K}}} \dfrac{\|A(x)\|}{\|x\|} = 0,$
(2) *$T(\mathbb{K}) \subseteq \mathbb{K}$ (i.e. T is positive),*
(3) *T has an asymptotic derivative $T_{\mathbb{K}}^{\infty}$ along the cone $\mathbb{K}$,*
(4) $\left\| T_{\mathbb{K}}^{\infty} \right\| < 1,$
(5) *$\mathrm{P}_{\mathbb{K}}[A(x) + T(x)]$ is scalarly compact,*

then the problem $NCP(f, \mathbb{K})$ has a solution.

Proof (Sketch only) We consider the mapping $\Phi : \mathbb{K} \to \mathbb{K}$ defined by $\Phi(x) = \mathrm{P}_{\mathbb{K}}[x - f(x)]$. Because $T_{\mathbb{K}}^{\infty}(\mathbb{K}) \subseteq \mathbb{K}$ we can show that $T_{\mathbb{K}}^{\infty}$ is also the asymptotic derivative of the mapping Φ and the assumptions of Theorem 18 are satisfied. We apply Theorem 18 and we obtain that Φ has a fixed point, which implies that the problem $NCP(f, \mathbb{K})$ has a solution. For more details about this proof, the reader is referred to [17]. □

Remark 8 We note that Theorem 19 has also interesting applications to complementarity theory. □

6.1 Open subjects

From the point of view of applications of Complementarity Theory to Economics and especially to Engineering, a very important subject is to find for complementarity problems *existence theorems for nontrivial solutions*. Before we define our opens subject we recall the following classical result given for Hilbert spaces.

Theorem 21 (Krasnoselskii) *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a normal closed convex cone and $f : H \to H$ a nonlinear completely continuous mapping such that $f(H) \subseteq$*

$\mathbb{K}$ *and* $f(0) = 0$. *Suppose that* f *has a Fréchet derivative* $f'(0)$ *at 0 and an asymptotic derivative* $f_{\mathbb{K}}^{\infty}$ *(both derivatives taken with respect to* $\mathbb{K}$*). If the following assumptions are verified:*

(1) $f_{\mathbb{K}}^{\infty}$ *has no eigenvalue* $\lambda \geqslant 1$,
(2) $f'(0)$ *has no positive eigenvector with corresponding eigenvalue* $\lambda_0 > 1$,

then f *has a non-zero fixed point in* $\mathbb{K}$.

Remark 9 We note that Theorem 21 is valid for k-set contractions with $0 \leq k < 1$ [9].

□

Open subjects:

(1) It is interesting to know if Theorem 21 is valid replacing the Fréchet derivative by a directional derivative along the cone $\mathbb{K}$. If this fact is true we can obtain an interesting existence theorem for nontrivial solutions for nonlinear complementarity problems.
(2) It seems to be interesting to apply Theorem 18 to differential or integral equations.

7 Quasi-bounded operators and complementarity problems depending of parameters

The notion of *quasi-bounded* operator is due to A. Granas [10]. This notion has been used in fixed point theory but never systematically in complementarity theory. First, we recall this notion.

Definition 11 Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be Banach spaces and $f : E \to F$ a mapping. We say that f is quasi-bounded if

$$\limsup_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|} = \inf_{\rho>0,\ \|x\|\geq\rho} \sup \frac{\|f(x)\|}{\|x\|} < +\infty.$$

Remark 10 We can show that $f : E \to F$ is quasi-bounded if and only if there exist $\rho > 0$ and a constant $M > 0$ such that $\|f(x)\| \leq M\|x\|$ for all x with $\|x\| \geq \rho$. □

Definition 12 If $f : E \to F$ is quasi-bounded, then the number

$$\|f\|_{qb} := \limsup_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|}$$

is called the quasi-norm of f.

Remark 11 If $T : E \to F$ is a bounded linear operator then T is quasi-bounded and we can prove that $\|T\|_{qb} = \|T\| = \sup_{\|x\|=1} \|T(x)\|$. □

The quasi-norm of a quasi-bounded operator has the following properties:

(1) If $f : E \to E$ is quasi-bounded then λf is quasi-bounded and we have $\|\lambda f\|_{qb} = |\lambda|\,\|f\|_{qb}$ for any $\lambda \in \mathbb{R}$.
(2) If $f_1, f_2 : E \to F$ are quasi-bounded, then $f_1 + f_2$ is quasi-bounded and we have that $\|f_1 + f_2\|_{qb} \leq \|f_1\|_{qb} + \|f_2\|_{qb}$.

(3) If $f : E \to F$ is quasi-bounded and $g : F \to G$ is linear and bounded, then $g \circ f$ is quasi-bounded.

Example 6

(1) If $f : E \to F$ is a mapping such that there exist $M_0 > 0$ and $M_1 \geq 0$ with the property that $\| f(x)\| \leq M_0 \|x\| + M_1$, for any $x \in E$, then f is quasi-bounded.
(2) If $f : E \to F$ is a mapping such that there exist $\rho_*, M_* > 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to+\infty} \frac{\varphi(t)}{t} < +\infty$ and $\| f(x)\| \leq M_* \|x\| + \varphi(\|x\|)$ for any $x \in E$, with $\|x\| \geq \rho_*$, then f is quasi-bounded. (This class of mappings contains the φ-bounded operators studied in nonlinear spectral analysis)
(3) If $f : E \to F$ has an asymptotic derivative, in the sense that there exists $S \in \mathcal{L}(E, F)$ such that $\lim_{\|x\|\to\infty} \frac{\|f(x)-S(x)\|}{\|x\|} = 0$, (equivalently: $f_\infty = S$), then f is quasi-bounded and $\|f\|_{qb} = \|f_\infty\| = \|S\|$.
(4) If f is ψ-additive with $\psi \in \mathcal{P}(\psi)$ and $f : E \to E$ is a mapping such that $f(\overline{B}(0, 1))$ is bounded, then f is quasi-bounded.

□

Remark 12 Because any asymptotically derivable operator is quasi-bounded we have interesting applications of quasi-bounded operators to the study of complementarity problems defined by integral operators and depending of parameters. □

Let $(H, \langle\cdot, \cdot\rangle)$ be a Hilbert space, $\mathbb{K} \subset H$ a closed convex cone ant $T, T_1, T_2 : H \to H$ mappings. We consider the mappings:

(1) $f_\rho(x) = \rho x - T(x)$, for all $x \in H$ and $\rho > 0$, $(\rho \in \mathbb{R})$,
(2) $f_{\rho,\lambda}(x) = \rho x - T_1(x) - \lambda T_2(x)$, for all $x \in H$ and $\lambda > 0$, $\rho > 0$ with $\lambda, \rho \in \mathbb{R}$,

and the complementarity problems $NCP(f_\rho, \mathbb{K})$ and $NCP(f_{\rho,\lambda}, \mathbb{K})$.

We say that $\rho > 0$ is an eigenvalue for the problem $NCP(f_\rho, \mathbb{K})$ if $NCP(f_\rho, \mathbb{K})$ has a solution $x_* \neq 0$. Similarly, for a given $\rho > 0$ we say that $\lambda > 0$ is an eigenvalue for $NCP(f_{\rho,\lambda}, \mathbb{K})$ if $NCP(f_{\rho,\lambda}, \mathbb{K})$ has a solution $x_* \neq 0$. We have the following results.

Theorem 22 *If $T : H \to H$ is quasi-bounded, completely continuous operator such that $\|T\|_{qb} < \rho$, then the problem $NCP(f_\rho, \mathbb{K})$ has a solution.*

Proof The proof is strongly based on the classical Leray-Schauder alternative and it is given in [19]. □

Corollary 4 *If $T : H \to H$ is a quasi-bounded completely continuous operator and if $T(0) \neq 0$, then any real number $\rho > \|T\|_{qb}$ is an eigenvalue for the problem $NCP(f_\rho, \mathbb{K})$. In particular, if T is asymptotically derivable and completely continuous, then any $\rho > \|T_\infty\|$ is an eigenvalue for the problem $NCP(f_\rho, \mathbb{K})$ if $T(0) \neq 0$*

Theorem 23 *IF $T_1 : h \to H$ is a linear completely continuous operator and $T_2 : H \to H$ is quasi-bounded and completely continuous, with the property that $T_2(0) \neq 0$, then for a given $\rho > 0$ there exists $\lambda_0 > 0$ such that any $\lambda \in]0, \lambda_0[$ is an eigenvalue to the problem $NCP(f_{\rho,\lambda}, \mathbb{K})$.*

Proof The proof is based on Theorem 22 and it is given in [19]. □

7.1 Open subjects

(1) The results presented in the is section support the idea of necessity to study deeply the class of quasi-bounded operators and to put in evidence new examples of quasi-bounded operators, especially related to practical problems.
(2) It is known that complementarity problems with eigenvalues are considered in elasticity and in the study of bifurcation problems for variational inequalities defined on cones (see the book by Vy Khoi Le and Klaus Schmitt [32])[5–8].
(3) A complementarity problem used in the study of equilibrium post-critic of this elastic plates is the complementarity problem defined by the Von-Karman operator, i.e. the operator of the from $f(x) = x - \lambda L(x) + T(x)$, where L is a linear completely continuous, self-adjoint operator, defined on a Hilbert space, $\lambda \in \mathbb{R}_+$ and T is a nonlinear completely continuous operator, homogeneous of degree three [5–8].

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